Few new reals

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The problem of building models of consequences, at the level of $H(\omega_2)$, of classical forcing axioms together with CH has a long history, starting with Jensen's landmark result that Suslin's Hypothesis is compatible with GCH.

Most of the work in the area done so far proceeds by showing that some suitable countable support iteration whose iterands are proper forcing notions not adding new reals fails to add new reals at limit stages.

There are (nontrivial) limitations to what can be achieved in this area (A–Larson–Moore): Modulo a mild large cardinal assumption, there are two Π_2 statements over $H(\omega_2)$, each of which can be forced, using proper forcing, to hold together with CH, and whose conjunction implies $2^{\aleph_0} = 2^{\aleph_1}$.

Above result closely tied to the following concrete well–known obstacle to not adding reals: Given a ladder system $\vec{C} = (C_\delta \,:\, \delta \in Lim(\omega_1))$, let $\text{Unif}(\vec{C})$ denote the statement that for every colouring $F : \text{Lim}(\omega_1) \longrightarrow \{0, 1\}$ there is $G : \omega_1 \longrightarrow \{0, 1\}$ such that that for every $\delta \in \text{Lim}(\omega_1)$ there is some $\alpha < \delta$ such that $G(\xi) = F(\delta)$ for all $\xi \in C_\delta \setminus \alpha$. We say that G uniformizes F on $\vec{C}$.

Given $\vec{C}$ and F as above there is a natural forcing notion, $\mathcal{Q}_{\vec{C},F}$, for adding a uniformizing function for F on $\vec{C}$ by initial segments. Easy to see that $\mathcal{Q}_{\vec{C},F}$ is proper, adds the intended uniformizing function, and does not add reals. However, any long enough iteration of forcings of the form $\mathcal{Q}_{\vec{C},F}$, even with a fixed $\vec{C}$, will necessarily add new reals. In fact, the existence of a ladder system $\vec{C}$ for which $\text{Unif}(\vec{C})$ holds cannot be forced together with CH in any way whatsoever, as this statement actually implies $2^{\aleph_0} = 2^{\aleph_1}$.

Proof: Fix a bijection $h : \omega \longrightarrow \omega \times \omega$ such that $i \leq n$ if $h(n+1) = (i, j)$. For each $g : \omega_1 \longrightarrow 2$ construct $f_n : \omega_1 \longrightarrow 2$ $(n < \omega)$ such that

$$f_0 = g$$

and

$$f_{n+1} \restriction C_\delta =_{\text{fin}} f_i(\delta + j)$$

for every limit $\delta \neq 0$, where $h(n+1) = (i, j)$.
Given f_k $(k \leq n)$, f_{n+1} exists by applying $\text{Unif}(\vec{C})$ to the colouring

$$\delta \longrightarrow f_i(\delta + j)$$

But now, for each limit $\delta \neq 0$, $(f_n \restriction \delta \;:\; n < \omega)$ determines $(f_n \restriction \delta + \omega \;:\; n < \omega)$. Hence,

$$(f_n \restriction \omega \;:\; n < \omega)$$

determines

$$(f_n \;:\; n < \omega),$$

and in particular $f_0 = g$. Hence $2^{\aleph_0} = 2^{\aleph_1}$. $\square$

In this work, we distance ourselves from the tradition of preserving CH by not adding reals; we aim at building interesting models of CH by a cardinal–preserving forcing which actually adds reals (but only $\aleph_1$–many of them).

Forcing with symmetric systems of models as side conditions

Finite–support forcing iterations involving symmetric systems of models as side conditions are naturally useful in situations in which, for example, we want to force

- consequences of classical forcing axioms at the level of $H(\omega_2)$, together with
- $2^{\aleph_0}$ large.

The pure side condition forcing

$$\mathcal{P}_0 = (\{\mathcal{N} : \mathcal{N} \text{ a } T\text{–symmetric system}\}, \supseteq)$$

(for any fixed $T \subseteq H(\kappa)$) preserves CH:

This exploits the fact that given N, $N' \in \mathcal{N}$, $\mathcal{N}$ a symmetric system, if $N \cap \omega_1 = N' \cap \omega_1$, then there is an isomorphism

$$\Psi : (N; \in, \mathcal{N} \cap N) \longrightarrow (N'; \in, \mathcal{N} \cap N')$$

(Of course there is at most one isomorphism between $(N; \in)$ and $(N'; \in)$.)

Proof sketch: Suppose $(\dot{r}_\xi)_{\xi<\omega_2}$ are names for subsets of ω and $\mathcal{N} \Vdash_{\mathcal{P}_0} \dot{r}_\xi \neq \dot{r}_{\xi'}$ for all $\xi \neq \xi'$. For each ξ, let N_ξ be a sufficiently correct model such that $\mathcal{N}$, $\dot{r}_\xi \in N_\xi$.

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By CH we may find $\xi \neq \xi'$ such that there is an isomorphism

$$\Psi : (N_\xi; \in, T^*, \mathcal{N}, \dot{r}_\xi) \longrightarrow (N_{\xi'}; \in, T^*, \mathcal{N}, \dot{r}_{\xi'})$$

(where T^* is the satisfaction predicate for $(H(\kappa); \in, T)$). Then $\mathcal{N}^* = \mathcal{N} \cup \{N_\xi, N_{\xi'}\} \in \mathcal{P}_0$. But $\mathcal{N}^*$ is $(N_\xi, \mathcal{P}_0)$–generic and $(N_{\xi'}, \mathcal{P}_0)$–generic.

Now, let $n < \omega$ and let $\mathcal{N}'$ be an extension of $\mathcal{N}^*$. Suppose $\mathcal{N}' \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. Then there is $\mathcal{N}'' \in \mathcal{P}_0$ extending both $\mathcal{N}'$ and some $\mathcal{M} \in N_\xi \cap \mathcal{P}_0$ such that $\mathcal{M} \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. By symmetry, $\mathcal{N}''$ extends also $\Psi(\mathcal{M})$. But $\Psi(\mathcal{M}) \Vdash_{\mathcal{P}_0} n \in \Psi(\dot{r}_\xi) = \dot{r}_{\xi'}$.

We have shown $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_\xi \subseteq \dot{r}_{\xi'}$, and similarly we can show $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_{\xi'} \subseteq \dot{r}_\xi$. Contradiction since $\mathcal{N}^*$ extends $\mathcal{N}$ and $\xi \neq \xi'$. $\square$

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In typical forcing iterations with symmetric systems as side conditions, $2^{\aleph_0}$ is large in the final extension. Even if $\mathcal{P}_0$ can be seen as the first stage of these iterations, the forcing is in fact designed to add reals at (all) subsequent successor stages.

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In this work we implement this idea. Our test problem: Measuring

Definition

Measuring holds if and only if for every sequence $\vec{C} = (C_\delta : \delta \in \omega_1)$, if each C_δ is a closed subset of δ in the order topology, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq C_\delta$, or
- $(C \setminus \alpha) \cap C_\delta = \emptyset$.

We say that C *measures* $\vec{C}$.

Natural forcing for adding a club measuring a given $\vec{C}$ by initial segments is proper and adds no new reals. On the other hand it is not known if these forcings can (consistently) be iterated without adding new reals. Strongest failures of Club–Guessing known to be within reach of current techniques for iterating proper forcing without adding reals are in the region of $\neg$ WCG (Shelah, NNR revisited).

Measuring implies $\neg$ WCG: Suppose $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ ladder system and $C \subseteq \omega_1$ is a club measuring $\vec{C}$. Then, for every $\delta \in C$, if δ is a limit point of limit points of C, then a tail of $C \cap \delta$ is disjoint from C_δ since $\mathrm{ot}(C_\delta) = \omega$.

Question

(J. Moore) Is Measuring consistent with CH?

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Our main theorem:

Theorem

(CH) Let $\kappa > \omega_2$ be a regular cardinal such that $2^{<\kappa} = \kappa$. There is then a partial order $\mathcal{P}$ with the following properties.

(1) *$\mathcal{P}$ is proper and $\aleph_2$–Knaster.*

(2) *$\mathcal{P}$ forces the following statements.*

   (a) *Measuring*
   (b) *CH*
   (c) *$2^\mu = \kappa$ for every uncountable cardinal $\mu < \kappa$.*

Our proof of this theorem does not apply to the case $\kappa = \omega_2$. This case is addressed by the following corollary.

Corollary

There is a partial order forcing the following statements.

1. *Measuring*
2. *CH*
3. $2^{\aleph_1} = \aleph_2$.

Proof.

Start with a model of CH and $2^{\aleph_2} = \aleph_3$ and let $\mathcal{P}$ be as in the Theorem with $\kappa = \omega_3$. Then, in $V^{\mathcal{P}}$, force with the collapse of ω_3 to ω_2 with conditions of cardinality $\aleph_1$. In the final model, $2^{\aleph_1} = \aleph_2$, and both CH and Measuring still hold since no new subsets of ω_1 have been added over $V^{\mathcal{P}}$. □

The construction

Let $\Phi : \{\alpha < \kappa \,:\, \mathrm{cf}(\alpha) = \omega_1\} \longrightarrow H(\kappa)$ be such that $\Phi^{-1}(x)$ is stationary in $\{\alpha < \kappa \,:\, \mathrm{cf}(\alpha) = \omega_1\}$ for all $x \in H(\kappa)$.
Let $(\theta_\alpha)_{\alpha<\kappa}$ be some fast enough increasing sequence of cardinals above $\beth_2(\kappa)$. For each $\alpha < \kappa$ let $\mathcal{M}^*_\alpha$ be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing Φ and $(\theta_\beta)_{\beta<\alpha}$, and let $\mathcal{M}_\alpha = \{N^* \cap H(\kappa) \,:\, N^* \in \mathcal{M}^*_\alpha\}$. Let T_α be a canonically chosen $T \subseteq H(\kappa)$ such that for every $N \in [H(\kappa)]^{\aleph_0}$, if $(N; \in, T) \preccurlyeq (H(\kappa); \in, T)$, then $N \in \mathcal{M}_\alpha$. Let also

$$\mathcal{T}_\alpha = \{N \in [H(\kappa)]^{\aleph_0} \,:\, (N; \in, T_\alpha) \preccurlyeq (H(\kappa); \in, T_\alpha)\}$$

and

$$\vec{T}_\alpha = \{(a, \xi) \in H(\kappa) \times \alpha + 1 \,:\, a \in T_\xi\}$$

Let $\beta < \kappa$ and suppose that $\mathcal{P}_\alpha$ has been defined for every $\alpha < \beta$. An ordered pair $q = (F, \Delta)$ is a $\mathcal{P}_\beta$–condition if and only if it has the following properties.

(1) F is a finite function such that $\mathrm{dom}(F) \subseteq \{\alpha < \beta : \mathrm{cf}(\alpha) = \omega_1\}$ and such that $F(\alpha)$ is a triple $(\mathcal{I}, b, \mathcal{O})$ for every $\alpha \in \mathrm{dom}(F)$.

(2) Δ is a finite functional collection of models with markers up to β.

(3) If $\mathrm{cf}(\beta) > \omega_1$ and $(N, \gamma) \in \Delta$, then $\gamma < \beta$.

(4) $\mathcal{N}^q_\beta$ is a closed $\vec{T}_\beta$–symmetric system.

(5) For every $\alpha < \beta$, the restriction of q to α,

$$q|_\alpha := (F \restriction \alpha, \Delta|_\alpha),$$

is a condition in $\mathcal{P}_\alpha$.

(6) Suppose $\beta = \alpha_0 + 1$ and $\mathrm{cf}(\alpha_0) = \omega_1$. Let $\dot{C}^{\alpha_0}$ be a canonically chosen $\mathcal{P}_{\alpha_0}$–name for a club–sequence on ω_1^V such that $\mathcal{P}_{\alpha_0}$ forces that

- $\dot{C}^{\alpha_0} = \Phi(\alpha_0)$ in case $\Phi(\alpha_0)$ is a $\mathcal{P}_{\alpha_0}$–name for a club–sequence on ω_1, and that
- $\dot{C}^{\alpha_0}$ is some fixed club–sequence on ω_1 in the other case.

If $\alpha_0 \in \mathrm{dom}(F)$, then $F(\alpha_0) = (\mathcal{I}^q_{\alpha_0}, b^q_{\alpha_0}, \mathcal{O}^q_{\alpha_0})$ has the following properties.

(a) $\mathcal{I}^q_{\alpha_0}$ is a finite set of pairwise disjoint intervals of the form $[\delta_0, \delta_1]$ for $\delta_0 \leq \delta_1 < \omega_1$.
(b) $\mathcal{O}^q_{\alpha_0} \subseteq \mathcal{N}^{q|\alpha_0}_{\alpha_0}$ is a $\vec{T}_{\alpha_0+1}$–symmetric system.
(c) $\mathcal{N}^q_{\alpha_0+1} \subseteq \mathcal{O}^q_{\alpha_0}$
(d) $\{\min(I) \,:\, I \in \mathcal{I}^q_{\alpha_0}\} = \{\delta_N \,:\, N \in \mathcal{N}^q_{\alpha_0+1}\}$
(e) $b^q_{\alpha_0}$ is a function such that

$$\mathrm{dom}(b^q_{\alpha_0}) \subseteq \{\min(I) \,:\, I \in \mathcal{I}^q_{\alpha_0}\}$$

and such that $b^q_{\alpha_0}(\delta) < \delta$ for every $\delta \in \mathrm{dom}(b^q_{\alpha_0})$.

(5) (continued)

(f) For all $\delta \in \mathrm{dom}(b^q_{\alpha_0})$ and all $I \in \mathcal{I}^q_{\alpha_0}$ such that $b^q_{\alpha_0}(\delta) < \min(I) < \delta$, there is some $r \in \mathcal{P}_{\alpha_0}$ such that $q|_{\alpha_0}$ extends r and such that

$$r \Vdash_{\mathcal{P}_{\alpha_0}} \min(I) \notin \dot{C}^{\alpha_0}(\delta),$$

where $\dot{C}^{\alpha_0}(\delta)$ is the canonical $\mathcal{P}_{\alpha_0}$–name for the member of $\dot{C}^{\alpha_0}$ indexed by δ.
Furthermore, if there is some $Q \in \mathcal{N}^q_\beta$ such that $\delta < \delta_Q$, then there is an r as above such that $r \in Q$ for some such Q of minimal height.

(g) Suppose $\alpha_0 \in \mathrm{dom}(F)$, $N \in \mathcal{N}^q_\beta$, and $\delta_N \in \mathrm{dom}(b^q_{\alpha_0})$. Then there is some $r \in \mathcal{P}_{\alpha_0}$ such that $q|_{\alpha_0}$ extends r and such that r forces in $\mathcal{P}_{\alpha_0}$ that exactly one of the following holds.

(i) There is some $a \in N$ for which there is no $M \in \mathcal{N}^{\dot{G}_{\alpha_0}}_{\alpha_0} \cap \mathcal{T}_\beta \cap N$ such that $a \in M$.

(ii) For every $a \in N$ there is some $M \in \mathcal{N}^{\dot{G}_{\alpha_0}}_{\alpha_0} \cap \mathcal{T}_\beta \cap N$ such that $a \in M$ and $\delta_M \notin \dot{C}^{\alpha_0}(\delta_N)$.

Furthermore, if there is some $Q \in \mathcal{N}^q_\beta$ such that $\delta_N < \delta_Q$, then there is an r as above such that $r \in Q$ for some such Q of minimal height.

(7) Suppose $(N_0, \gamma_0), (N_1, \gamma_1) \in \Delta$, $\delta_{N_0} = \delta_{N_1}$, and $\gamma = \min\{\gamma_0, \gamma_1\}$. Let

$$\alpha_0 = \sup\{\xi \in N_0 \cap \gamma \ : \ \Psi_{N_0,N_1}(\xi) < \gamma\}$$

and

$$\alpha_1 = \sup\{\xi \in N_1 \cap \gamma \ : \ \Psi_{N_1,N_0}(\xi) < \gamma\}$$

Then then there is some $n < \omega$ such that

$$n = |\operatorname{dom}(F) \cap N_0 \cap \alpha_0| = |\operatorname{dom}(F) \cap N_1 \cap \alpha_1|;$$

furthermore, letting $(\xi_i^0)_{i<n}$ and $(\xi_i^1)_{i<n}$ be the strictly increasing enumerations of $\operatorname{dom}(F) \cap N_0 \cap \alpha_0$ and $\operatorname{dom}(F) \cap N_1 \cap \alpha_1$, respectively, Ψ_{N_0,N_1} is an isomorphism between the structures

$$(N_0; \in, \Phi, \vec{T}_{\alpha_0}, \Delta|_{\alpha_0}, \mathcal{I}^q_{\xi_i^0}, b^q_{\xi_i^0}, \mathcal{O}^q_{\xi_i^0})_{i<n}$$

and

$$(N_1; \in, \Phi, \vec{T}_{\alpha_1}, \Delta|_{\alpha_1}, \mathcal{I}^q_{\xi_i^1}, b^q_{\xi_i^1}, \mathcal{O}^q_{\xi_i^1})_{i<n}$$

Given $\mathcal{P}_\beta$–conditions q_i, for $i = 0, 1$, let us say that q_1 extends q_0 if and only if the following holds.

(1) For every $(N, \gamma) \in \Delta_{q_0}$ there is some $\gamma' \geq \gamma$ such that $(N, \gamma') \in \Delta_{q_1}$.

(2) The following holds for every $\alpha \in \mathrm{dom}(F_{q_0})$.

  (a) For every $I \in \mathcal{I}_\alpha^{q_0}$ there is some $I' \in \mathcal{I}_\alpha^{q_1}$ such that $I \subseteq I'$ and $\min(I) = \min(I')$.

  (b) $b_\alpha^{q_0} \subseteq b_\alpha^{q_1}$

  (c) $\mathcal{O}_\alpha^{q_0} \subseteq \mathcal{O}_\alpha^{q_1}$

Finally, we let $\mathcal{P}_\kappa = \bigcup_{\beta<\kappa} \mathcal{P}_\beta$ and for all $\beta < \kappa$ and $q_0, q_1 \in \mathcal{P}_\beta$, we say that q_1 extends q_0 in $\mathcal{P}_\kappa$ if and only if q_1 extends q_0 in $\mathcal{P}_\beta$.

The construction is not to be regarded as an iteration: Suppose $q \in \mathcal{P}_\beta$, $\alpha < \beta$, D is a dense subset of $\mathcal{P}_\alpha$, and we want to find $r \in D$ compatible with q. Suppose, for example, that we have (N, γ) and (N', γ') in Δ_q, $\alpha_1 \in N \cap \gamma \cap N' \cap \gamma'$, $\alpha_0 \in N \cap \alpha_1$, $\Psi_{N,N'}(\alpha_0) = \alpha_0'$, $\alpha_0 < \alpha \leq \alpha_0'$, and $\delta_N = \delta_{N'} \in \text{dom}(b^q_{\alpha_0'})$. If we are not careful enough when picking r in D – and there seems to be no reason to expect that we can always be careful enough –, then r could be, for example, such that $F_r(\alpha_0)$ contains intervals I with $b^q_{\alpha_0'}(\delta_N) < \min(I) < \delta_N$ and such that $q|_{\alpha_0'}$ happens to force $\min(I) \in \dot{C}^{\alpha_0'}(\delta_{N'})$, which would make it impossible to amalgamate q and r into a condition in $\mathcal{P}_\beta$.

On the other hand,

- $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_{\alpha+1}$, and
- there is an ω_1–club $C \subseteq \kappa$ such that $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$ for all $\alpha < \beta$ in $C \cup \{\kappa\}$.

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On the other hand,

- $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_{\alpha+1}$, and
- there is an ω_1–club $C \subseteq \kappa$ such that $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$ for all $\alpha < \beta$ in $C \cup \{\kappa\}$.

Some open questions

Not hard to see that Measuring is equivalent to: If $(\mathcal{C}_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is such that each $\mathcal{C}_\delta$ is a collection of closed subsets of δ such that $|\mathcal{C}_\delta| \leq \aleph_0$, then there is a club of ω_1 measuring all members of $\mathcal{C}_\delta$ for each δ.

We may thus consider the following family of strengthenings of Measuring.

Definition

Given a cardinal κ, Meas_κ holds if and only if for every family $\mathcal{C}$ consisting of closed subsets of ω_1 and such that $|\mathcal{C}| \leq \kappa$ there is a club $C \subseteq \omega_1$ with the property that for every $D \in \mathcal{C}$ and every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq D$, or
- $((C \cap \delta) \setminus \alpha) \cap D = \emptyset$.

$\mathsf{Meas}_{\aleph_0}$ is trivially true in ZFC. Also, it is clear that Meas_κ implies Meas_λ whenever $\lambda < \kappa$, and that $\mathsf{Meas}_{\aleph_1}$ implies Measuring.

The splitting number, $\mathfrak{s}$, is the minimal cardinality of a splitting family, i.e., of a collection $\mathcal{X} \subseteq [\omega]^{\aleph_0}$ such that for every $Y \in [\omega]^{\aleph_0}$ there is some $X \in \mathcal{X}$ such that $X \cap Y$ and $Y \setminus X$ are both infinite.

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Fact

$\text{Meas}_{\mathfrak{s}}$ is false.

Proof.

Let $\mathcal{X} \subseteq [\omega]^{\aleph_0}$ be a splitting family. Let $(C_\delta)_{\delta \in \text{Lim}(\omega)}$ be a ladder system on ω_1 and let $\mathcal{C}$ be the collection of all sets of the form

$$Z_\delta^X = \bigcup\{[C_\delta(n),\, C_\delta(n+1)] \,:\, n \in X\} \cup \{\delta\}$$

for some $\delta \in \text{Lim}(\omega_1)$ and $X \in \mathcal{X}$. Let D be a club of ω_1, let $\delta < \omega_1$ be a limit point of D, and let

$$Y = \{n < \omega \,:\, [C_\delta(n),\, C_\delta(n+1)] \cap D \neq \emptyset\}$$

Let $X \in \mathcal{X}$ such that $X \cap Y$ and $Y \setminus X$ are infinite. Then $Z_\delta^X \cap D$ and $D \setminus Z_\delta^X$ are both cofinal in δ. Hence, D does not measure $\mathcal{C}$. □

Similarly, Krueger has observed that $\text{Meas}_{\mathfrak{a}}$ is false.

The following question is open.

Question

Is $\mathit{Meas}_{\aleph_1}$ *consistent?*

Conjecture: $\mathrm{Meas}_{\aleph_1}$ follows from BPFA.

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A further extension of Measuring:

Measuring is of course equivalent to the same statement replacing closed subsets C_δ of δ (for $\delta \in \mathrm{Lim}(\omega_1)$) by open sets. A natural next step would be to consider arbitrary countable unions of closed subsets of δ. This is of course the same thing as an arbitrary subsets of δ. This leads to the following:

Definition

Measuring*: For every $(X_\delta \,:\, \delta \in \mathrm{Lim}(\omega_1))$, if $X_\delta \subseteq \delta$ for all δ, then there is a club $C \subseteq \omega_1$ such that for all $\delta \in C$, a tail of $C \cap \delta$ is either contained in or disjoint from X_δ.

This is the strongest conceivable failure of Club Guessing.

Fact

*(*ZFC*) Measuring** *is false.*

Proof.

Let $S \subseteq \omega_1$ be stationary and co-stationary, and let $X_\delta = S \cap \delta$ for all $\delta \in \mathrm{Lim}(\omega_1)$. Then there is no club C measuring X_δ for all $\delta \in C$. □

Fortunately, the status of Measuring* is more interesting under $\neg$AC:

Fact

(ZF+ The club filter on ω_1, $\mathcal{C}_{\omega_1}$, is a normal filter on ω_1) Suppose $\vec{X} = (X_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is such that

(1) *$X_\delta \subseteq \delta$ for each δ.*

(2) *For each club $C \subseteq \omega_1$,*

  (a) *there is some $\delta \in C$ such that $C \cap X_\delta \neq \emptyset$, and*

  (b) *there is some $\delta \in C$ such that $(C \cap \delta) \setminus X_\delta \neq \emptyset$.*

Then there is a stationary and co-stationary subset of ω_1 definable from $\vec{X}$.

Proof.

Case 1: For all $\alpha < \omega_1$, either

- $W^0_\alpha = \{\delta < \omega_1 : \alpha \notin X_\delta\}$ is in $\mathcal{C}_{\omega_1}$, or
- $W^1_\alpha = \{\delta < \omega_1 : \alpha \in X_\delta\}$ is in $\mathcal{C}_{\omega_1}$.

For each $\alpha < \omega_1$ let W_α be W^ϵ_α for the unique $\epsilon \in \{0, 1\}$ such that $W^\epsilon_\alpha \in \mathcal{C}_{\omega_1}$, and let $W^* = \Delta_{\alpha<\omega_1} W_\alpha \in \mathcal{C}_{\omega_1}$. Then $X_{\delta_0} = X_{\delta_1} \cap \delta_0$ for all $\delta_0 < \delta_1$ in W^*. It then follows, by (2), that $S = \bigcup_{\delta \in W^*} X_\delta$, which of course is definable from $\vec{C}$, is a stationary and co-stationary subset of ω_1.

Case 2: There is some $\alpha < \omega_1$ with the property that both W^0_α and W^1_α are stationary subsets of ω_1. But now we can let S be W^0_α, where α is first such that W^0_α is stationary and co-stationary. □

It is worth comparing the above observation with Solovay's classic result that an ω_1–sequence of pairwise disjoint stationary subsets of ω_1 is definable from any given ladder system on ω_1 (working in the same theory).

Corollary

(ZF + The club filter on ω_1, $\mathcal{C}_{\omega_1}$, is normal) The following are equivalent:

1. *Measuring**
2. *$\mathcal{C}_{\omega_1}$ is an ultrafilter.*
3. *For every sequence $(X_\delta : \delta \in \mathrm{Lim}(\omega_1))$, if $X_\delta \subseteq \delta$ for each δ, then there is a club $C \subseteq \omega_1$ such that either*
   - *$C \cap \delta \subseteq X_\delta$ for every $\delta \in C$, or*
   - *$C \cap X_\delta = \emptyset$ for every $\delta \in C$.*

In particular, AD implies Measuring*.

Question

(ZF + $\mathcal{C}_{\omega_1}$ is normal). Suppose $\mathcal{C}_{\omega_1}$ is not an ultrafilter. Are there more than two disjoint stationary subsets of ω_1?

Back in ZFC, the following question, suggested by Moore, aims at addressing the issue whether or not adding new reals is a necessary feature of any successful approach to forcing Measuring + CH.

Question

Does Measuring imply that there are non-constructible reals?

Bonus track: Trees on $\aleph_2$ and GCH

This is joint work in progress with Mohammad Golshani.

As mentioned at the beginning, Jensen proved the consistency of Suslin's Hypothesis with GCH.

A natural question (appearing in the literature since at least 1978 (Kanamori–Magidor)):

Question

Does GCH imply the existence of an $\aleph_2$–Suslin tree?

Laver–Shelah (1981), proved the consistency of CH + "Every Every $\aleph_2$–Aronszajn tree is special", starting from weakly compact cardinal κ. Their construction is of the form $\mathrm{Coll}(\omega_1, <\kappa) * \dot{Q}$, where $\dot{Q}$ is κ–c.c. in $V^{\mathrm{Coll}(\omega_1, <\kappa)}$.

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Assuming the existence a strongly unfoldable cardinal, in recent joint work with Golshani we have constructed a poset that should force GCH + Every $\aleph_2$–Aronszajn tree is special.

Forcing can be roughly describes as: Force a suitable symmetric system $\mathcal{N} \subseteq ([H(\kappa^{++})]^{<\kappa})^V$. Then do the Laver–Shelah construction incorporating 'twin models with markers' (N_0, γ_0), (N_1, γ_1), where N_0, $N_1 \in \mathcal{N}$, between which we require to have strong symmetry, as in the measuring construction, in order to enforce preservation of $2^{\aleph_1} = \aleph_2$.

The poset is σ–closed. It adds new subsets of ω_1, but only $\aleph_2$–many of them.

Most likely, a weakly compact cardinal is enough. That would give an equiconsistency.

We are currently writing up the details and everything seems fine.